Survey of magnetic monopoles in the massless limit

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(Dated: June 18, 2013)

The massless limit of magnetic monopole is obtained through Lorentz boost to the speed of light. The stationary monopole is taken to be the 't Hooft-Polyakov monopole of SU(2) gauge group and triplet of real Higgs fields. Using lightcone coordinates and Lorentz transformation rules for the four-momentum, it is shown that no finite energy solutions of topological solitons in one direction with the speed of light exists. This excludes the possibility of a magnetic monopole moving with the speed of light having finite energy.

Since Dirac discussed the possibility of explaining the quantisation of electric charge from the existence of a magnetic monopole [1] its very existence became a subject worthy of interest to many physicists. As of yet, however, no single experiment turned out to be conclusive on their existence, and this rareness is usually attributed to their relatively heavy mass, the reason why much of current research consider them massive. On the other hand, studies on monopoles without rest mass are rather uncommon. Empirical evidence aside, can massless magnetic monopoles exist in theory?

Let us probe how far classical electromagnetism can take us. It is known that when a charged particle is moving at high speeds the electromagnetic field induced by the charge tends to appear compressed in the direction of motion. In the limiting case of the speed of light, the fields would be concentrated on the plane orthogonal to the moving direction. The solution to

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m \ ,$$

becomes(in cylindrical coordinates)

$$\mathbf{B} = \frac{2e_m}{\rho}\delta(ct-z)\hat{\rho} \ , \tag{1}$$

or using potentials

$$\begin{aligned} \phi &= 0 \\ \mathbf{A} &= \frac{2e_m u(ct-z)}{\rho}\hat{\phi} \ , \end{aligned} \tag{2}$$

where $u(x)$ is the Heaviside step function. Interestingly after the monopole has past an electric charge the vector potential gives that charge a constant kinetic angular momentum

$$K_z = \rho(p_\phi - \frac{e_e}{c}A_\phi) \ .$$

As kinetic angular momentum is the quantity that is quantized in quantum theory, the monopole strength is quantized by the condition equivalent to that found by Dirac [1].

$$\frac{2e_e e_m}{c\hbar} = n$$

Unfortunately the vector potential found becomes singular at the negative z-axis, which is just the Dirac string in ultrarelativistic case. Fortunately, this singular property can be eliminated totally by using a nonabelian gauge theory; the 't Hooft-Polyakov monopole.

't Hooft-Polyakov monopole is a stationary topological soliton which behaves as a magnetic monopole. The following is based on the derivation by Valery Rubakov [2]. Natural units($c = 1$) are used. Consider the Georgi-Glashow model with SU(2) gauge group with the Lagrangian of real fields given as;

$$\mathfrak{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2}D_\mu\varphi^a D_\mu\varphi^a - \frac{\lambda}{4}\left(\varphi^a\varphi^a - v^2\right)^2 \tag{3}$$

$$D_\mu\varphi^a = \partial_\mu\varphi^a + g\epsilon^{abc}A^b_\mu\varphi^c \tag{4}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu \tag{5}$$

The greek indices run from 0 to 3 and latin indices run from 1 to 3. The vacuum value of the Higgs field is $\varphi^a\varphi^a = v^2$ with gauge freedom. The popular gauge choice of small perturbations for the ground state, $\varphi = (0, 0, v + \eta)$, fixes the freedom and is usually referred to as the unitary gauge. Using the unitary gauge one can determine the masses of the massive scalar field η and two vector fields A^1_μ, A^2_μ as

$$\begin{aligned} m_H &= \sqrt{2\lambda}v \\ m_V &= gv \end{aligned}$$

while the other vector field A^3_μ remains massless. The massless vector field corresponds to the electromagnetic vector field. The equation of motion for this model is

$$(D_\mu F_{\mu\nu})^a + g\epsilon^{abc}(D_\nu\varphi)^b\varphi^c = 0 \tag{6}$$

$$(D_\mu D_\mu\varphi)^a + \lambda(v^2 - \varphi^b\varphi^b)\varphi^a = 0 \tag{7}$$

A significant feature of this model is that a stationary configuration of the Higgs field that is different from vacuum can exist. An example is a solution with the boundary condition for the Higgs field at spatial infinity given as vacuum value oriented radially, i.e. asymptotic form of the Higgs field far from the origin given as

$$\varphi = \frac{v}{r}(x, y, z),\ r \equiv \sqrt{x^2 + y^2 + z^2}$$

This is an example of a topological soliton. The finite energy solution with this boundary condition could be found by using the ansatz

$$\begin{aligned} \varphi^a &= n^a v H(r) && (8)\\ A_i^a &= \frac{1}{gr}\epsilon^{aij} n^j (1 - F(r)) && (9)\\ A_0^a &= 0 \end{aligned}$$

where the vector $\mathbf{n} = \frac{\mathbf{x}}{r}$ is the unit radius vector. The boundary conditions are given by

$$r \to 0 : H \to 0, F \to 1 \qquad (10)$$
$$r \to \infty : H \to 1, F \to 0 \qquad (11)$$

Though the solution has to be sought numerically the limit $\lambda \to 0$, which is called the Bogomolny-Prasad-Sommerfield limit, has an exact solution.

$$\begin{aligned} F &= \frac{\xi}{\sinh \xi} && (12)\\ H &= \coth \xi - \frac{1}{\xi} && (13)\\ \xi &= gvr \end{aligned}$$

The electromagnetic field is the massless mode of $F^a_{\mu\nu}$, which lies along the direction of φ^a. In gauge-invariant form this is written as

$$\mathcal{F}_{\mu\nu} = \frac{1}{v}\varphi^a F^a_{\mu\nu}$$

The electromagnetic fields, $\mathcal{E}_i$ electric and $\mathcal{B}_i$ magnetic, take the form of

$$\begin{aligned} \mathcal{E}_i &= \mathcal{F}_{0i} = 0 && (14)\\ \mathcal{B}_i &= -\frac{1}{2}\epsilon^{ijk}\mathcal{F}_{jk} = \frac{1-F^2}{gr^2} n^i && (15) \end{aligned}$$

and applying the boundary condition (11) it is evident that the magnetic field far away from the origin follows the inverse-square law which is expected of magnetic monopoles. By doing a Lorentz boost on the solutions, the massless kinematic limit is obtained.

Lorentz boost in z-direction will be used throughout this paper. Using rapidity α, defined $\alpha = \tanh^{-1}(v/c)$, the Lorentz boost is written as

$$\begin{aligned} t' &= t\cosh\alpha + z \sinh\alpha \\ x' &= x \\ y' &= y \\ z' &= z\cosh\alpha + t\sinh\alpha \end{aligned} \qquad (16)$$

Evidently a boost by the speed of light runs into infinities as the rapidity for the speed of light is infinity. One has to adopt a different method to circumvent this pitfall. Comparing (2) with the potential similar to that Dirac used in his analysis (given by Sakurai [3]) an insight could be provided.

$$\mathbf{A}_{\text{stat}} = \frac{e_m(1-\cos\theta)}{r\sin\theta}\hat{\phi} \qquad (17)$$

Under scrutiny, it could be found that the substitution of $r = \rho|\cot\theta|$ and taking the limit [4]

$$\lim_{\theta\to 0} \mathbf{A}_{\text{stat}} \quad z - ct > 0$$
$$\lim_{\theta\to \pi} \mathbf{A}_{\text{stat}} \quad z - ct < 0$$

derives (2) from (17). One may proceed to take the same limit for the 't Hooft-Polyakov monopole, but this method would only work for vector components perpendicular to the direction of motion; not only that the vector component parallel to the direction of motion changes magnitude (as can be verified from (16)) but also an additional constraint exists: The integral of it is invariant under Lorentz boosts. If the time component of a static four-vector field $A_\mu(x)$ is zero,

$$\int_{-\infty}^{\infty} A_z(x,y,z)dz = \int_{-\infty}^{\infty} A_{z'}(x',y',z',t')dz' \qquad (18)$$

holds. This can be proved from the transformation rules for the z-component of $A_\mu(x)$.

$$A_{z'}(x',y',z',t') = \cosh\alpha\, A_z(x',y',z'\cosh\alpha - t'\sinh\alpha)$$

Similar conservation rule holds for tensors.

$$\int_{-\infty}^{\infty} F_{zx}(x,y,z)dz = \int_{-\infty}^{\infty} F_{z'x'}(x',y',z',t')dz' \text{ , etc.} \qquad (19)$$

The fields vanish at spatial infinity, and any small deviation of the difference of the transformed coordinates $z' - t'$ would correspond to spatial infinity in the stationary coordinates $txyz$ at the limit $\alpha \to \infty$. Therefore vectors and tensors are nonzero only at the equator of motion $z' - t' = 0$ and the components that are parallel to motion should have a Dirac-delta form. The strengths of the deltas are calculated from (18) or (19), respectively. The values of components perpendicular to the direction of motion are given by the values at some $z = \text{const}$.

Unfortunately even the exact solutions (12) and (13) need numerical integration to give the strengths of the deltas. The result (FIG. 1) shows that for sufficiently large ξ the 't Hooft-Polyakov monopole (15) becomes indistinguishable to the classical solution (1). The method discussed does not seem to contradict expectations.

Will the total energy diverge to infinity? The solution at Bogomolny-Prasad-Sommerfield limit is known to have stationary energy of [2]

$$\begin{aligned} M_{\lambda=0} &= \frac{4\pi v}{g} = \frac{m_V}{\alpha_g} \\ \alpha_g &= \frac{g^2}{4\pi} \end{aligned}$$

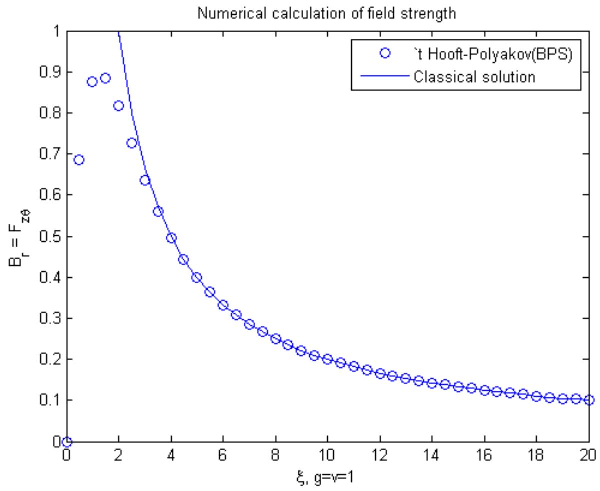

FIG. 1: Numerical calculation of 't Hooft-Polyakov monopole(BPS limit) boosted to the speed of light

and for general case of $\lambda \neq 0$ stationary energy is finite as well [2]. It could be inferred that 't Hooft-Polyakov monopole boosted to the speed of light has infinite energies, which could also be shown from energy density containing squares of Dirac deltas.

The Lorentz boost of the static solution to the speed of light inevitably requires the energy of the monopole to be infinite. Is a solution with finite energy with the speed of light permissible? Because it is impossible to put such solutions, if they ever exist, in a reference frame where they are stationary, a dynamic description is necessary. However as the solution moves along a direction with constant speed, use of lightcone coordinates reduces the problem to three-dimensions; if the coordinates are chosen such that

$$\begin{aligned} v &= \frac{1}{\sqrt{2}}(z+t) \\ u &= \frac{1}{\sqrt{2}}(z-t) \end{aligned}$$

then a field configuration moving along the positive z axis would be independent of the coordinates v, thereby leaving the coordinates x, y, and u as the independent variables. In this coordinate system the metric tensor is given as

$$\begin{aligned} g_{uv} = g_{vu} = g_{xx} = g_{yy} = 1 \\ g_{\mu\nu} = 0 \ \text{ otherwise.} \end{aligned}$$

The solution with the following properties will be sought after;

1. The solution obeys Lorentz transformation.
2. The form of the solution remains unchanged as it moves.
3. The square of the total momentum of the configuration is zero.

The first of the properties is essential for a solution that is compatible with special relativity, so no explanation seems necessary. The second requires the solution to be stable and do not dissipate as time passes. This follows from the requirement that the solution propagates in only one direction. The last property is needed to ensure that the signs of energy and momentum do not change under Lorentz boosts along the moving direction.

It is easy to verify that a Lorentz boost in $vuxy$ coordinates along the z-direction is given as dilatations in v and u coordinates;

$$\begin{array}{rcl} v' &=& e^{\alpha} v \\ u' &=& e^{-\alpha} u \\ x' &=& x \\ y' &=& y \end{array} \Leftrightarrow \begin{array}{rcl} t' &=& t\cosh\alpha + z\sinh\alpha \\ x' &=& x \\ y' &=& y \\ z' &=& z\cosh\alpha + t\sinh\alpha \end{array} \tag{20}$$

For future reference, the transformation of the four-momentum of systems with zero rest mass is given.

$$\begin{aligned} p^{\mu} = (E,0,0,p) &= (p,0,0,p) \\ p' &= e^{\alpha} p \end{aligned} \tag{21}$$

Now think of the stress-energy tensor of the fields. From a Lagrangian density, one can obtain a symmetric stress-energy tensor by varying the Lagrangian with respect to the metric tensor.

$$-\frac{1}{2}\sqrt{-g}T_{\mu\nu} = \frac{\delta(\sqrt{-g}\mathfrak{L})}{\delta g^{\mu\nu}}$$

The energy density is the time-time component in $txyz$ coordinates, and the total energy of the system is obtained by integrating it over the entire space.

$$E = p^0 = \int d^3x\, T^{tt} = \int d^3x\, T_{tt} \tag{22}$$

The equation (22) is a numerical relationship so the upper and lower indices do not need to match.

The coordinate transformations relate the stress-energy tensor of different coordinates. The time-time component of the stress-energy tensor in $txyz$ coordinates expressed in components of the $vuxy$ coordinates is

$$T_{tt} = \frac{1}{2}\left(T_{vv} + T_{uu} - 2T_{uv}\right) \ . \tag{23}$$

Inserting (23) into (22),

$$E = \frac{1}{2}\int d^3x\, \left(T_{vv} + T_{uu} - 2T_{uv}\right) \ . \tag{24}$$

Lorentz boost transforms each components in the following way

$$\begin{array}{rcl} E &\to& e^{\alpha} E \\ T_{vv} &\to& e^{-2\alpha} T_{vv} \\ T_{uu} &\to& e^{2\alpha} T_{uu} \\ T_{uv} &\to& T_{uv} \\ d^3x &\to& e^{-\alpha} d^3x \end{array}$$

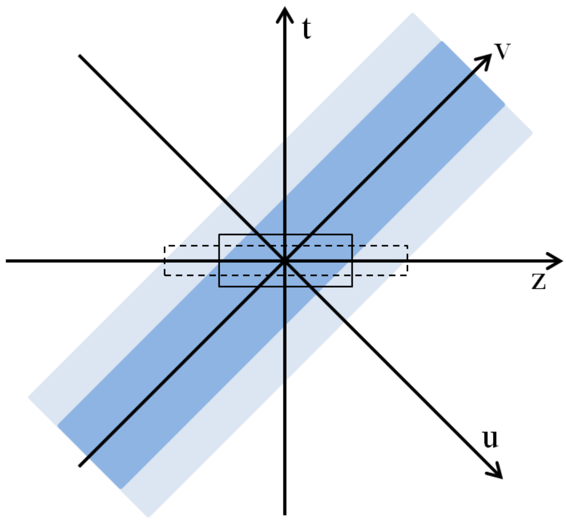

FIG. 2: $u-v$ axis superposed on $t-z$ axis

Although the others directly follow from the transformation rules, the transformation of the spatial volume element is not so evident. This can be understood by visualization using the $t-z$ diagram (FIG. 2) ; for $\alpha > 0$, this Lorentz boost contracts the field configurations from the lightly shaded region to the darker regions [5]. Therefore the integration domain shrinks from the region enclosed by dashed lines to the region enclosed by solid lines. This diagram shows that the spatial volume element is dilated by the same factor as the u coordinates, which is precisely $e^{-\alpha}$.

The transformation rule (21) requires the following components of the stress-energy tensor to be zero.

$$T_{vv} = T_{uv} = 0 \tag{25}$$

For the Lagrangian being considered, the stress-energy tensor in its explicit form is

$$T_{\mu\nu} = F^a_{\mu\rho} F^a_{\nu\rho} + D_\mu \varphi^a D_\nu \varphi^a - g_{\mu\nu} \mathfrak{L} \tag{26}$$

Applying the equation (25) and taking account of nonzero components of the metric tensor in $vuxy$ coordinates yields

$$T_{vv} = 0 \;\Rightarrow\; F^a_{v\mu} = D_v \varphi^a = 0 \tag{27}$$

$$\begin{array}{rl} F^a_{v\mu} = D_v\varphi^a & = 0 \\ T_{uv} & = 0 \end{array} \;\Rightarrow\; \mathfrak{L} = 0 \tag{28}$$

The above conditions force the Lagrangian density to depend only on x and y coordinates as components with u coordinate index are always cancelled out by the zero values of the components with v coordinate index. All components of the Lagrangian density are enumerated.

$$\begin{aligned} \mathfrak{L} &= -\frac{1}{2}\left(F^a_{xy}F^a_{xy} + D_x\varphi^a D_x\varphi^a + D_y\varphi^a D_y\varphi^a\right) \\ &\quad -\frac{\lambda}{4}\left(\varphi^a\varphi^a - v^2\right)^2 \\ &= 0 \end{aligned} \tag{29}$$

Since every term in (29) is real and all terms enter as absolute squares with the same sign, all terms must be zero; the Higgs field has vacuum expectations. Outside the integration domain the fields are equivalent to vacuum and as Higgs field cannot possess other values than vacuum inside the integration domain, the Higgs field has vacuum values everywhere. This excludes the possibility of a topological soliton with aforementioned properties, thereby excluding the possibility of a finite energy monopole with the speed of light.

Although the quantization condition provided by classical electromagnetism tempts one to look for magnetic monopoles moving with the speed of light, it has been argued that magnetic monopoles with zero rest mass and finite energy cannot exist, at least for generally accepted form of Lagrangian densities. Unless a different form of Lagrangian is used, consideration of magnetic monopoles as massless entities seem unnecessary.

It should be mentioned that the condition (29) seems to imply that all solutions possessing the three properties must take a plane-wave form; changes of field components in the direction normal to the direction of the motion must be zero. This could be generalized further to apply to $n+1$-dimensions with a Lagrangian of the form (3) as well. It is conjectured that topological solitions formed from the Higgs field cannot have a plane-wave form as covariant derivatives and equation of motion restrict the distribution of the Higgs field to be exponentially reliant on u-coordinates [6], which is unsuitable for topological solitons. If this is true, topological solitons cannot be massless in general. On the other hand, addition of total derivatives to the Lagrangian does not alter the equation of motion of the system and this possibility has not been surveyed. These questions remain to be answered in the future.

I acknowledge Prof. Rey, Soo-Jong for valuable advice and suggestions, and Prof. Lee, Sangmin for constructive comments for this paper.

[1] P.A.M. Dirac, **Proc. Roy. Soc.**, A**133** (1934), p. 60
[2] Valery Rubakov, *Classical Theory of Gauge Fields* (Princeton University Press, New Jersey, USA, 2002), Chapter 9
[3] J.J. Sakurai, *Modern Quantum Mechanics*, Revised Ed. (Addison Wesley, USA, 1994), page 141
[4] This equation contains c only for legibility.
[5] As the configuration is independent of v-coordinates, the dilatation of v-coordinates is insignificant.
[6] The conjecture supposes the Higgs field to be a fundamental representation of the gauge group, which is different from the representation considered in this paper.