

Frobenius Theorem in General Relativity

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This is a commentary to section 2.3.3 *Frobenius' theorem* of Eric Poisson's book *A Relativist's Toolkit*. The physical interpretation of hypersurface orthogonal geodesic congruences and its relation to irrotationality (in the sense that rotation tensor $\omega_{\alpha\beta}$ vanishes) will be discussed. It will be shown that this is related to the well-known paradox of special relativity; "what is the ratio of circumference to diameter of a rotating disk?". This manuscript has tried to keep discussion of equations to a minimum, so one who is interested in detailed computations must consult other references.

1 A Paradox of Special Relativity; Circumference of a Rotating Disk

1.1 The paradox

Consider a rotating rigid¹ disk of diameter D . Standard geometric arguments say that the circumference of this disk should be πD . On the other hand, because this disk is rotating the circumference of this disk must undergo length contraction so that its circumference becomes $\gamma^{-1}\pi D$. The two statements cannot be true simultaneously unless $\gamma = 1/\sqrt{1-v^2} = 1$, i.e. the disk is *not* rotating. Where is the loophole?

This discussion is somewhat irrelevant to the usual argument of why rigidity is forbidden in relativity, so if you think that the use of the word "rigid" is the loophole then consider a slightly modified version of this paradox. Consider N number of particles that can move along a circular track of diameter D . The particles now start to move, maintaining their relative distances to neighbouring particles. Because the particles maintained their relative distances, the sum of relative distances should be the same to the sum evaluated when particles were at rest. Why didn't length contraction occur?² Can you spot the loophole?³

1.2 The resolution

Let's come back to the derivation of length contraction. The notion of (three dimensional) length is obviously not well defined in four dimensions, so how did we *define* length when discussing length contraction?

¹Rigidity is a nonsensical word in special relativity, but I have chosen this word for the sake of simplicity.

²To be honest, this statement is wrong; it is impossible to retain relative distances while accelerating. The closest thing one can do is to adjust the relative distances so that the distances *increase* by the Lorentz factor γ (in the large N limit).

³By the way, some references report that this paradox was one of the main problems Einstein had in mind when he developed general relativity[3].

To make a long story short, we have implicitly chosen a moment of the observer who observes length contraction. In other words, a *time slice* of the reference frame we are interested in was chosen when we derived the formula for length contraction. This time slice then has an induced metric (inherited from the original spacetime) which we can use to measure the length of an object.

This procedure of defining length is the loophole of the paradox mentioned in the previous section. To measure length, we need the notion of *equal time slice* of the spacetime continuum. To have foliation of spacetime by equal time slices, we need some kind of procedure to *synchronise clocks* on different points. To have a more concrete visualisation, let's put infinitesimal clocks whose worldlines does not collide into another on different points of the spacetime continuum and think of spacetime continuum as a collection of them. This means each clock and its time reading defines a unique point of the spacetime continuum. Given this collection of infinitesimal clocks, *is it possible to consistently synchronise⁴ the readings of all clocks?*

The answer: No. The paradox (of the second version) is busted by the fact that it is impossible to synchronise clocks attached to the particles consistently. Picking a reference particle and synchronising the clock readings of the particle that succeeds it, one finds that after going around the whole cycle one must synchronise the reference particle's clock to the reference particle's clock *of the past*. This is the reason why naïve analysis of the rotating disk problem seems to give inconsistent answers.

2 Frobenius Theorem in General Relativity

2.1 Frobenius' Theorem of Geodesic Congruences

A congruence is a collection of curves in some open region \mathcal{O} of spacetime where for every point in \mathcal{O} a curve from the collection which passes through the point is uniquely determined. You will notice that this mathematical object is motivated from the collection of infinitesimal clocks that we have considered in the previous section. When all curves of a collection are (time-like) geodesics, the congruence is called a (time-like) geodesic congruence. The behaviour of a (time-like) geodesic congruence is determined by the tensor $B_{\alpha\beta} = u_{\alpha;\beta}$, which can be decomposed in the following way.

$$B_{\alpha\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \quad (1)$$

$$\theta = B_{\alpha}^{\alpha} \quad \text{expansion scalar} \quad (2)$$

$$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta} \quad \text{shear tensor} \quad (3)$$

$$\omega_{\alpha\beta} = B_{[\alpha\beta]} \quad \text{rotation tensor} \quad (4)$$

The tensor $h_{\alpha\beta}$ is the transverse metric defined by the relation $g_{\alpha\beta} = h_{\alpha\beta} - u_{\alpha}u_{\beta}$.

Frobenius' theorem states that for (time-like) geodesic congruences with vanishing rotation tensor $\omega_{\alpha\beta} = 0$ one can find a family of space-like hypersurfaces which is everywhere orthogonal to

⁴The definition of consistent synchronisation will be given in a later part of this manuscript.

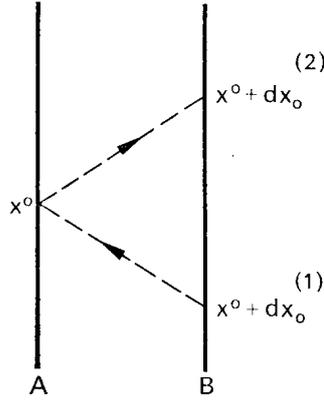


그림 1: Synchronising clocks. Figure taken from Landau and Lifshitz vol. 2.

the congruence. The converse also works, which can be shown by a simple calculation; a geodesic congruence orthogonal to a family of space-like hypersurfaces is irrotational.

$$\omega_{\alpha\beta} = 0 \iff \exists \Phi(x^\alpha) \text{ s.t. } u_\alpha \propto -\Phi_{,\alpha} \quad (5)$$

The hypersurfaces are defined by the relation $\Phi(x^\alpha) = \text{const}$. In fact, it can be shown that by appropriate reparametrisation the proportionality factor can be put to 1. The proof is given in the book *A Relativist's Toolkit* so we omit its proof here.

$$\omega_{\alpha\beta} = 0 \iff \exists \Psi(x^\alpha) \text{ s.t. } u_\alpha = -\Psi_{,\alpha} \quad (6)$$

In the following sections we will look for the theorem's implications to physics, centered on its relation to the paradox of special relativity that has been elaborated at the beginning of this manuscript.

2.2 Synchronisation of Clocks in General Relativity and the Meaning of the Hypersurface

In this section we formalise the idea of *consistent synchronisation*. Let's consider the same question in absence of gravity; how do we synchronise clock in special relativity? Recalling the memory of general physics 101 (although the section on relativity is likely to have been hastily glossed over, leaving no trace of anything at all), we used *light signals* to synchronise clocks.

1. Pick a reference clock and a clock to synchronise with this reference clock. The two clocks should be relatively stationary to each other.
2. Send a light signal (call it A) at $t_{ref} = a$ from the reference clock to the clock being synchronised.
3. Upon receiving the signal A, the clock being synchronised immediately sends another light signal (call it B) to the reference clock.

4. The reference clock receives the signal B at $t_{ref} = b$.
5. Readjust the clock reading of the clock being synchronised so that the time read at the moment of reception of signal A becomes $t'_{sync} = (a + b)/2$.

We can do the same thing in presence of gravitation. The details of this discussion will be relegated to section “*Distances and time intervals*” of Landau and Lifshitz volume 2, *The Classical Theory of Fields*, and we just state the conclusion here: Consistent clock synchronisation can be done in reference systems with metric components $g_{0i} = 0$. A coordinate system with metric components $g_{0i} = 0$ and $g_{00} = -1$ is called a *synchronous reference system*, the subject which the same book also allocates a section under the title *The synchronous reference system*.

The meaning of the function $\Psi(x^\alpha)$ that defines the hypersurfaces $\Psi(x^\alpha) = c$ is the reading of synchronised clocks at that point. This is most clearly shown when we consider a coordinate system that uses the value of $y^0 = \Psi$ as the time coordinate and other three functions y^1 , y^2 , and y^3 to specify a point on the hypersurface $\Psi(x^\alpha) = c$. In this system, the metric components obey the condition $g_{0i} = 0$ and $g_{00} = -1$; it is a synchronous reference system. This clarifies the physical reason why we cannot have non-vanishing rotation tensor for hypersurface orthogonal time-like geodesic congruences; relatively rotating clocks cannot be consistently synchronised, so if clocks can be consistently synchronised then the clocks must not be rotating.

3 References

1. E. Poisson, *A Relativist's Toolkit*
2. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, A Course of Theoretical Physics vol. 2
3. Stachel, John (1980). “*Einstein and the Rigidly Rotating Disk*”. In A. Held. *General Relativity and Gravitation*. quoted in Wikipedia article *Ehrenfest paradox*